Worcester Polytechnic Institute January, 2015

Linear Algebra General Comprehensive Exam

Print Name: ____________________________ Sign: ____________________________

In the following, $\mathbb{R}^{\mathrm{n}}$ is real n-dimensional space, $\mathbb{C}^{\mathrm{n}}$ is complex n-dimensional space, and $\mathbb{R}^{\mathrm{n\times n}}$ is the space of real $n \times n$ matrices. If no specific basis is mentioned then all linear transformations are expressed through the canonical basis.

1. Let $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

and let $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

a. For both A and B compute the rank, the determinant, the characteristic polynomial, eigenvalues and eigenvectors.

b. Find the transformation matrix M and its inverse such that $J = M^{-1}AM$ is the Jordan canonical form of A.

c. Find the transformation matrix N and its inverse such that $J = N^{-1}BN$ is the Jordan canonical form of B.

d. Are A and B similar?

2. There are two equally large containers separated by a soluble membrane. In one container there is a salt solution with concentration of 5 parts per million, while the other contains a salt solution with concentration of 55 parts per million. The membrane dissolves.

Let $a(t)$ denote the salt concentration in container A at time t, and $b(t)$ be the salt concentration in container B.

We have $\dfrac{da}{dt} = b - a$ and $\dfrac{db}{dt} = a - b$, with $a(0) = 5$ and $b(0) = 55$.

a) Find the matrix M such that $\dfrac{d\mathbf{x}}{dt} = M\mathbf{x}$ with $\mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix}$.

b) Find eigenvalues and eigenvectors of M.

c) Is the salt concentration in container A increasing?

d) What can you say about the limit, as n goes to infinity of M^n?

e) What can you say about the limit, as t goes to infinity of $a(t)$ and $b(t)$?

3. Prove that the polynomials $p(x)$ of degree at most 3 satisfying $p(0) = p(1/2) = p(1) = 0$ form a subspace of the vector space of all polynomials of degree at most 3 on $[0, 1]$ and find a basis for this subspace.

a) Is this subspace invariant with respect to the linear transformation represented in the canonical basis by $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

b) Is this subspace invariant w.r.t. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$?

c) Compute the orthogonal projection of x^3 onto this subspace (with respect to the inner product $\langle p_1, p_2 \rangle = \int_{-1}^{1} p_1(x)p_2(x)\,dx$).

4. Consider the linear system $Ax = b$, where $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$.

a. Discuss the existence and uniqueness of solutions of this system. (Does a solution exist for every b? If not, then for what b, if any, does a solution exist? If a solution exists for some b, is it unique? If not, describe all possible solutions.)

b. Find a least-squares solution for $b = (1 \quad 0 \quad 1 \quad 0)^T$ and discuss its existence and uniqueness.

5. Consider a real coefficients $n \times n$ matrix $A = (a_{ij})_{i,j=1,\ldots,n}$. The determinant $det(A)$ is a function of the coefficients $a_{1,1}, \ldots., a_{n,n}$. Show that this is a smooth function.

a. Compute the matrix of partial derivatives $\left(\partial_{a_{i,j}} det(A)\right)_{i,j=1,\ldots,n}$.

b. Use the previous part to compute $\frac{d}{dt} \det\left(e^{tA}\right)$. (You can assume A is invertible if you prefer). Can you use this result to easily find out an expression for $\det(e^{tA})$?

6. Find the subset V of the space of all real valued $n \times n$ matrices defined by

$$V = \{A \in \mathbb{R}^{n,n} \mid u(x) \text{ is harmonic} \Leftrightarrow u(Ax) \text{ is harmonic}\}.$$

(Reminder: a function is harmonic in $\mathbb{R}^n$ if $\sum_{i=1}^{n} \partial^2_{x_i} u(x) = 0$ for all $x \in \mathbb{R}^n$).

a. Compute all possible determinants of matrices in V.

b. Show that V is a group, i.e. (a) if $A, B \in V$ then $AB, BA \in V$; (b) there exists a *identity element* $C \in V$ such that $CA = AC = A$ for all $A \in V$; (c) for each $A \in V$ there exists $B \in V$ such that $AB = BA = C$ (the identity) and (d) the group multiplication is associative $(AB)D = A(BD)$ for all $A, B, D \in V$.